

Problem 5197. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 4$.

Prove that

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \leq \frac{1}{xyz}$$

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The Power Mean inequality yields

$$\frac{x+y+z}{3} \leq \sqrt{\frac{x^2+y^2+z^2}{3}} = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3} \Rightarrow x+y+z \leq 2\sqrt{3} \quad (1)$$

According to HM-AM inequality we have $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$, for every $a, b \in \mathbb{R}^+$, therefore

$$\begin{aligned} \frac{1}{6-x^2} &= \frac{1}{\frac{3}{2}(x^2+y^2+z^2)-x^2} = \frac{2}{2x^2+3y^2+3z^2} \\ &= \frac{1}{2} \cdot \frac{4}{(x^2+3y^2)+(x^2+3z^2)} \\ &\leq \frac{1}{2} \left(\frac{1}{x^2+3y^2} + \frac{1}{x^2+3z^2} \right) \end{aligned}$$

By AM-GM inequality $x^2+3y^2 \geq 2\sqrt{3}xy$ and $x^2+3z^2 \geq 2\sqrt{3}xz$, so

$$\frac{1}{6-x^2} \leq \frac{1}{2} \left(\frac{1}{2\sqrt{3}xy} + \frac{1}{2\sqrt{3}xz} \right) = \frac{1}{4\sqrt{3}} \left(\frac{1}{xy} + \frac{1}{xz} \right) \quad (2)$$

Analogously we get

$$\frac{1}{6-y^2} \leq \frac{1}{4\sqrt{3}} \left(\frac{1}{xy} + \frac{1}{yz} \right) \quad (3)$$

and

$$\frac{1}{6-z^2} \leq \frac{1}{4\sqrt{3}} \left(\frac{1}{yz} + \frac{1}{xz} \right) \quad (4)$$

Adding (2), (3) and (4) gives us

$$\begin{aligned} \frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} &\leq \frac{1}{4\sqrt{3}} \left(\frac{1}{xy} + \frac{1}{xz} + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{yz} + \frac{1}{xz} \right) \\ &= \frac{1}{2\sqrt{3}} \left(\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} \right) \\ &= \frac{1}{2\sqrt{3}} \cdot \frac{x+y+z}{xyz} \\ &\leq \frac{1}{xyz} \end{aligned}$$

where in the last step we used (1). The proof is complete. \square